

Asymptotic Expansion of Solutions to the Drift-Diffusion Equation with Fractional Dissipation

MASAKAZU YAMAMOTO¹ YUUSUKE SUGIYAMA²

ABSTRACT. The initial-value problem for the drift-diffusion equation arising from the model of semiconductor device simulations is studied. The dissipation on this equation is given by the fractional Laplacian $(-\Delta)^{\theta/2}$. Large-time behavior of solutions to the drift-diffusion equation with $0 < \theta \leq 1$ is discussed. When $\theta > 1$, large-time behavior of solutions is known. However, when $0 < \theta \leq 1$, the perturbation methods used in the preceding works would not work. Large-time behavior of solutions to the drift-diffusion equation with $0 < \theta \leq 1$ is discussed. Particularly, the asymptotic expansion of solutions with high-order is derived.

1. INTRODUCTION

We study the following initial-value problem for the drift-diffusion model for semiconductors:

$$(1.1) \quad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u - \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 2$, $0 < \theta < n$, $\partial_t = \partial/\partial t$, $\nabla = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $\Delta = \partial_1^2 + \dots + \partial_n^2$, and $(-\Delta)^{\theta/2} \varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]]$. The unknown functions u and $\psi : (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ stand for the density of electrons and the potential of electromagnetic field, respectively. The drift-diffusion equation with $\theta = 2$ is derived from conservation of mass of electrons. The fractional Laplacian is associated to the jumping process in the stochastic process. Since electrons on a semiconductor may jump from a dopant to another, the fractional Laplacian is suitable to describe their dissipation. In the case $\theta > 1$, well-posedness and global existence of solutions are shown. Moreover, large-time behavior of the solution is discussed (cf. [1, 23, 24, 26, 28, 33, 39]). When $\theta > 1$, we can refer to many preceding works to derive the asymptotic expansion of the solution of (1.1) as $t \rightarrow \infty$ (cf. for example [3, 7, 8, 12–14, 20, 29]). In this case, perturbation methods are effective, since the highest-order derivative is on the dissipation term. When $\theta = 1$, ∇u on the nonlinear term balances the dissipation $(-\Delta)^{1/2} u$. In the case $0 < \theta < 1$, the highest-order derivative is on the nonlinear term. Therefore, the perturbation methods would not work as discussed in more detail later. In [41], employing the energy method, the authors estimate the difference between the solution of (1.1) with $\theta = 1$ and its second-order asymptotic expansion in $L^q(\mathbb{R}^n)$ for $1 < q < \infty$. But the cases $q = 1$ and $q = \infty$ are excepted. The purpose of this paper is to give the third-order asymptotic expansion for (1.1) with $0 < \theta \leq 1$. Especially, we will estimate the difference between the solution and the asymptotic expansion in $L^q(\mathbb{R}^n)$ with $1 \leq q \leq \infty$. Our main theorems are extensions from the results of the case $1 < \theta \leq 2$ in [39] to $0 < \theta \leq 1$. For the drift-diffusion equation with $0 < \theta \leq 1$, we refer to the preceding works for the following two-dimensional quasi-geostrophic equation:

$$\begin{cases} \partial_t u + (-\Delta)^{\theta/2} u - \nabla^\perp \psi \cdot \nabla u = 0, & t > 0, \ x \in \mathbb{R}^2, \\ (-\Delta)^{1/2} \psi = u, & t > 0, \ x \in \mathbb{R}^2, \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. For the quasi-geostrophic equation, well-posedness, and global in time existence for small initial data in the scale-invariant Besov spaces is shown (cf. [4–6]). By analogy from the method in the quasi-geostrophic equation, for the drift-diffusion equation with $0 < \theta \leq 1$, well-posedness and global existence for small initial data in the scale-invariant Besov space are shown

¹Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan

²Department of Mathematics, Tokyo University of Science, Tokyo 162-8601, Japan

in [36]. In [36], global existence for positive initial data is also studied. We consider the solution such that

$$(1.2) \quad u \in L^\infty(0, \infty; L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)), \quad \|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p})}$$

for $1 \leq p \leq \infty$, and

$$(1.3) \quad u \in C^\infty((0, \infty), H^\infty(\mathbb{R}^n)).$$

In [2, 25, 36, 40], it is shown that solutions satisfy (1.2) and (1.3), if initial data are sufficiently smooth and nonnegative. Upon the above assumption, the conservative force fulfills that

$$(1.4) \quad \|\nabla \psi(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p})+\frac{1}{\theta}}$$

for $\frac{n}{n-1} < p \leq \infty$ (see Proposition 2.8 in Section 2). To discuss large-time behavior of the solution, we introduce the fundamental solution of $\partial_t u + (-\Delta)^{\theta/2} u = 0$:

$$G_\theta(t, x) = \mathcal{F}_\xi^{-1}[e^{-t|\xi|^\theta}](x).$$

In the case $\theta = 1$, this function equals to the Poisson kernel

$$P(t, x) = \pi^{-\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) t (t^2 + |x|^2)^{-\frac{n+1}{2}}.$$

The Duhamel formulae rewrites the solution of (1.1) by the mild solution as follows:

$$(1.5) \quad u(t) = G_\theta(t) * u_0 + \int_0^{t/2} \nabla G_\theta(t-s) * (u \nabla (-\Delta)^{-1} u)(s) ds + \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds.$$

We remark that, in the case $\theta > 1$, the second and the third terms are combined into $\int_0^t \nabla G_\theta(t-s) * (u \nabla (-\Delta)^{-1} u)(s) ds \in C([0, T], L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ since $\nabla G_\theta \in L^1(0, T, L^1(\mathbb{R}^n))$ and $u \nabla (-\Delta)^{-1} u \in L^\infty(0, T, L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$. But, if $\theta \leq 1$, $\nabla G_\theta \notin L^1(0, T, L^1(\mathbb{R}^n))$, which requires estimates for ∇u . Furthermore, the third-order asymptotic expansion needs some estimates for xu (see the remark after Theorem 1.1). However, (1.5) does not work in those estimates, since the third term of (1.5) contains ∇u . Employing the energy method with Kato and Ponce's commutator estimate and the positivity lemma for the fractional Laplacian, we get those estimate for ∇u and xu respectively (see Propositions 2.7 and 2.9). Our first assertion is established as follows.

Theorem 1.1. *Let $n = 3$ and $0 < \theta < 1$, or $n \geq 4$ and $0 < \theta \leq 1$. Assume that $u_0 \in L^1(\mathbb{R}^n, (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^n)$ and the solution u satisfies (1.2) and (1.3). Then*

$$\left\| u(t) - MG_\theta(t) - m \cdot \nabla G_\theta(t) - \sum_{|\alpha|=2} \frac{\nabla^\alpha G_\theta(t)}{\alpha!} \int_{\mathbb{R}^n} (-y)^\alpha u_0(y) dy \right. \\ \left. - \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t) \cdot \int_0^\infty \int_{\mathbb{R}^n} (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds \right\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$, where $M = \int_{\mathbb{R}^n} u_0(y) dy$ and $m = \int_{\mathbb{R}^n} (-y) u_0(y) dy$.

We remark that the decay properties of u ensure that $\int_{\mathbb{R}^n} |y| |u \nabla (-\Delta)^{-1} u| dy \leq C(1+s)^{-\frac{n-2}{\theta}}$ (see Proposition 2.9). Hence the coefficient $\int_0^\infty \int_{\mathbb{R}^n} (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds$ in Theorem 1.1 converges to a finite value since $\theta < n - 2$. However, if $\theta \geq n - 2$, this coefficient may diverge to infinity. In this case, we should include some correction terms in the asymptotic expansion. When $n = 2$ and $0 < \theta \leq 1$, let J be given by

$$(1.6) \quad J(t) = \int_0^{t/2} \nabla G_\theta(t-s) * (G_\theta \nabla (-\Delta)^{-1} G_\theta)(s) ds + \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (G_\theta \nabla (-\Delta)^{-1} G_\theta)(s) ds.$$

Then the same argument as in [38] yields that

$$J \in C((0, \infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)), \quad J \neq 0,$$

and

$$(1.7) \quad \|J(t)\|_{L^q(\mathbb{R}^2)} = t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2-\theta}{\theta}} \|J(1)\|_{L^q(\mathbb{R}^2)}$$

for $1 \leq q \leq \infty$. Moreover, J satisfies the following.

Theorem 1.2. *Let $n = 2$, $0 < \theta < 1$, $u_0 \in L^1(\mathbb{R}^2, (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^2)$, and J be given by (1.6). Assume that the solution u satisfies (1.2) and (1.3). Then*

$$\begin{aligned} & \left\| u(t) - MG_\theta(t) - m \cdot \nabla G_\theta(t) - M^2 J(t) - \sum_{|\alpha|=2} \frac{\nabla^\alpha G_\theta(t)}{\alpha!} \int_{\mathbb{R}^2} (-y)^\alpha u_0(y) dy \right. \\ & \quad \left. - \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t) \cdot \int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds \right\|_{L^q(\mathbb{R}^2)} \\ & = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}}) \end{aligned}$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$, where $M = \int_{\mathbb{R}^2} u_0(y) dy$ and $m = \int_{\mathbb{R}^2} (-y) u_0(y) dy$.

Before the proof of this theorem (see the remark under the proof of Proposition 3.2 in Section 3), we will confirm that

$$(1.8) \quad \int_0^\infty \int_{\mathbb{R}^2} |y| |u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta| dy ds < +\infty.$$

Unfortunately, when $n = 3$ and $\theta = 1$, the first term of J may diverge to infinity since $P \nabla(-\Delta)^{-1} P(s)$ is too singular as $s \rightarrow 0$. For this case, we define

$$\begin{aligned} (1.9) \quad \tilde{J}(t) &= \int_0^{t/2} \int_{\mathbb{R}^3} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \cdot (P \nabla(-\Delta)^{-1} P)(s, y) dy ds \\ &+ \int_{t/2}^t P(t-s) * \nabla \cdot (P \nabla(-\Delta)^{-1} P)(s) ds, \\ \tilde{K}(t) &= \frac{1}{3} \Delta P(t) \log(1 + \frac{t}{2}) \int_{\mathbb{R}^3} (-y) \cdot (P \nabla(-\Delta)^{-1} P)(1, y) dy. \end{aligned}$$

The function \tilde{J} fulfills

$$(1.10) \quad \tilde{J} \in C((0, \infty), L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \quad \|\tilde{J}(t)\|_{L^q(\mathbb{R}^3)} = t^{-3(1-\frac{1}{q})-2} \|\tilde{J}(1)\|_{L^q(\mathbb{R}^3)}$$

for $1 \leq q \leq \infty$ (see Proposition 3.5 in Section 3), and provides the asymptotic expansion for the solution as follows.

Theorem 1.3. *Let $n = 3$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^3, (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^3)$, and \tilde{J} and \tilde{K} be given by (1.9). Assume that the solution u satisfies (1.2) and (1.3). Then*

$$\begin{aligned} & \left\| u(t) - MP(t) - m \cdot \nabla P(t) - M^2 \tilde{K}(t) - M^2 \tilde{J}(t) - \sum_{|\alpha|=2} \frac{\nabla^\alpha P(t)}{\alpha!} \int_{\mathbb{R}^3} (-y)^\alpha u_0(y) dy \right. \\ & \quad \left. - \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^\infty \int_{\mathbb{R}^3} (-y)^\beta (u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(1+s, y)) dy ds \right\|_{L^q(\mathbb{R}^3)} \\ & = o(t^{-3(1-\frac{1}{q})-2}) \end{aligned}$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$, where $M = \int_{\mathbb{R}^3} u_0(y) dy$ and $m = \int_{\mathbb{R}^3} (-y) u_0(y) dy$.

If we try to give the asymptotic expansion for the case $n = 2$ and $\theta = 1$ in the same way as above, then we may see that $\int_0^\infty \int_{\mathbb{R}^2} |y| |u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P| dy ds = +\infty$. To study this case, we

define

$$\begin{aligned}
(1.11) \quad J_2(t) &= M \int_0^{t/2} \nabla P(t-s) * (P \nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P) \nabla(-\Delta)^{-1}P)(s) ds \\
&\quad + M \int_{t/2}^t P(t-s) * \nabla \cdot (P \nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P) \nabla(-\Delta)^{-1}P)(s) ds \\
&\quad + M^3 \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \\
&\quad \quad \cdot (P \nabla(-\Delta)^{-1}J + J \nabla(-\Delta)^{-1}P)(s, y) dy ds \\
&\quad + M^3 \int_{t/2}^t P(t-s) * \nabla (P \nabla(-\Delta)^{-1}J + J \nabla(-\Delta)^{-1}P)(s) ds, \\
K(t) &= \frac{1}{2} \Delta P(t) \log(1 + \frac{t}{2}) \int_{\mathbb{R}^2} (-y) \cdot (P \nabla(-\Delta)^{-1}J + J \nabla(-\Delta)^{-1}P)(1, y) dy.
\end{aligned}$$

Then J_2 satisfies

$$(1.12) \quad J_2 \in C((0, \infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)), \quad \|J_2(t)\|_{L^q(\mathbb{R}^2)} = t^{-2(1-\frac{1}{q})-2} \|J_2(1)\|_{L^q(\mathbb{R}^2)}$$

for $1 \leq q \leq \infty$ (see Proposition 3.6 in Section 3).

Theorem 1.4. *Let $n = 2$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^2, (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^2)$, and J , J_2 and K be given by (1.6) and (1.11). Assume that the solution u satisfies (1.2) and (1.3). Then*

$$\begin{aligned}
&\left\| u(t) - MP(t) - m \cdot \nabla P(t) - M^2 J(t) - M^3 K(t) - J_2(t) - \sum_{|\alpha|=2} \frac{\nabla^\alpha P(t)}{\alpha!} \int_{\mathbb{R}^2} (-y)^\alpha u_0(y) dy \right. \\
&\quad - \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\
&\quad \quad \left. - M (P \nabla(-\Delta)^{-1}(m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)(1 + s, y) \} dy ds \right\|_{L^q(\mathbb{R}^2)} \\
&= o(t^{-2(1-\frac{1}{q})-2})
\end{aligned}$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$, where $M = \int_{\mathbb{R}^2} u_0(y) dy$ and $m = \int_{\mathbb{R}^2} (-y) u_0(y) dy$.

We confirm that

$$\begin{aligned}
(1.13) \quad &\int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\
&\quad - M (P \nabla(-\Delta)^{-1}(m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)(1 + s, y) \} dy ds \in \mathbb{R}^2
\end{aligned}$$

in Section 3. Theorem 1.4 provides the asymptotic expansion with third-order. Clearly, we see that the asymptotic expansion with second-order contains no logarithmic term. Now we refer to the following generalized Burgers equation:

$$(1.14) \quad \begin{cases} \partial_t \omega + (-\partial_x^2)^{1/2} \omega + \frac{1}{2} \partial_x(\omega^2) = 0, & t > 0, \quad x \in \mathbb{R}, \\ \omega(0, x) = \omega_0(x), & x \in \mathbb{R}. \end{cases}$$

For (1.14), well-posedness, global existence and decay of solutions for small initial data are proved. Particularly, for $1 \leq q \leq \infty$, the decaying solution has the following asymptotic expansion as $t \rightarrow \infty$ (see [15, 41]):

$$\begin{aligned}
(1.15) \quad &\left\| \omega(t) - M_\omega P(t) + \frac{1}{4\pi} M_\omega^2 \partial_x P(t) \log(1 + \frac{t}{2}) - M_\omega^2 J_\omega(t) \right. \\
&\quad \left. - \left(m_\omega - \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} (\omega(s, y)^2 - M_\omega^2 P(1 + s, y)^2) dy ds \right) \partial_x P(t) \right\|_{L^q(\mathbb{R})} = o(t^{-(1-\frac{1}{q})-1}),
\end{aligned}$$

where $M_\omega = \int_{\mathbb{R}} \omega_0(y) dy$, $m_\omega = \int_{\mathbb{R}} (-y) \omega_0(y) dy$ and

$$J_\omega(t) = -\frac{1}{2} \int_0^{t/2} \int_{\mathbb{R}} (\partial_x P(t-s, x-y) - \partial_x P(t, x)) P(s, y)^2 dy ds \\ - \int_{t/2}^t P(t-s) * (P \partial_x P)(s) ds.$$

This correction term fulfills

$$\|J_\omega(t)\|_{L^q(\mathbb{R})} = t^{-(1-\frac{1}{q})-1} \|J_\omega(1)\|_{L^q(\mathbb{R})}$$

for $1 \leq q \leq \infty$. The logarithmic term in (1.15) is derived from the following procedure: The mild solution of (1.14) is given by

$$\omega(t) = P(t) * \omega_0 - \frac{1}{2} \int_0^{t/2} \partial_x P(t-s) * \omega(s)^2 ds - \int_{t/2}^t P(t-s) * (\omega \partial_x \omega)(s) ds.$$

In the second term, we renormalize ω by $M_\omega P$, then we obtain the term $\frac{1}{2} M_\omega^2 \int_0^{t/2} \partial_x P(t-s) * P(1+s)^2 ds$. Taylor's theorem says that the decay rate of this term is given by

$$\frac{1}{2} M_\omega^2 \partial_x P(t) \int_0^{t/2} \int_{\mathbb{R}} P(1+s, y)^2 dy ds = \frac{1}{2} M_\omega^2 \partial_x P(t) \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}} P(1, y)^2 dy \\ = \frac{1}{4\pi} M_\omega^2 \partial_x P(t) \log(1 + \frac{t}{2}).$$

Here we used the relation $P(1+s, y) = (1+s)^{-2} P(1, (1+s)^{-1}y)$. Similarly, the second-order asymptotic expansion for (1.1) with $n = 2$ and $\theta = 1$ contains

$$M^2 \nabla P(t) \cdot \int_0^{t/2} \int_{\mathbb{R}^2} (P \nabla (-\Delta)^{-1} P)(1+s, y) dy ds \\ = M^2 \nabla P(t) \log(1 + \frac{t}{2}) \cdot \int_{\mathbb{R}^2} (P \nabla (-\Delta)^{-1} P)(1, y) dy,$$

since $P(1+s, y) = (1+s)^{-3} P(1, (1+s)^{-1}y)$ when $n = 2$. This fact does not contradict the assertion of Theorem 1.4. Indeed

$$\int_{\mathbb{R}^2} (P \nabla (-\Delta)^{-1} P)(1, y) dy = 0.$$

Such a vanishing logarithmic term is developed in the studies for some other phenomena (we refer to [10, 11, 18, 21, 30–32, 37]).

Notation. In this paper, we use the following notation. For $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we denote that $a \cdot b = \sum_{j=1}^n a_j b_j$ and $|a| = \sqrt{a \cdot a}$. We define the Fourier transform and the Fourier inverse transform by $\mathcal{F}[\varphi](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx$ and $\mathcal{F}^{-1}[\varphi](x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(\xi) d\xi$, where $i = \sqrt{-1}$. We denote that $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ ($j = 1, \dots, n$), $\nabla = (\partial_1, \dots, \partial_n)$ and $\Delta = \sum_{j=1}^n \partial_j^2$. Particularly $\partial_x = \partial/\partial x$ for $n = 1$, and $\nabla^\perp = (-\partial_2, \partial_1)$ for $n = 2$. For $\theta > 0$, $(-\Delta)^{\theta/2} \varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]]$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n = (\mathbb{N} \cup \{0\})^n$, we use $\alpha! = \prod_{j=1}^n \alpha_j!$, $\nabla^\alpha = \prod_{j=1}^n \partial_j^{\alpha_j}$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, $L^p(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ denote the Lebesgue space and the Sobolev space on \mathbb{R}^n , respectively. We abbreviate the norm of $L^p(\mathbb{R}^n)$ by $\|\cdot\|_{L^p(\mathbb{R}^n)}$. For a nonnegative function g , let $L^1(\mathbb{R}^n, g dx) = \{\varphi \in L_{loc}^1(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\varphi(x)| g(x) dx < +\infty\}$. We write the convolution of $f = f(x)$ and $g = g(x)$ by $f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy$. The gamma function is provided by $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$ for $p > 0$. Various constants are simply denoted by C .

2. PRELIMINARIES

In this section, we prepare several lemmas to use in the proof of our results.

Lemma 2.1 (positivity lemma). *Let $0 \leq s \leq 2$, $p \geq 1$ and $f \in W^{s,p}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f|^{p-2} f (-\Delta)^{s/2} f dx \geq 0.$$

Particularly, when $p \geq 2$,

$$\int_{\mathbb{R}^n} |f|^{p-2} f (-\Delta)^{s/2} f dx \geq \frac{2}{p} \int_{\mathbb{R}^n} \left| (-\Delta)^{s/4} (|f|^{p/2}) \right|^2 dx$$

holds.

For the proof of this lemma, see [6, 16]. We also need some inequalities of Sobolev type.

Lemma 2.2 (Hardy-Littlewood-Sobolev's inequality [35, 42]). *Let $n \geq 2$, $1 < \sigma < n$, $1 < p < \frac{n}{\sigma}$ and $\frac{1}{p_*} = \frac{1}{p} - \frac{\sigma}{n}$. Then there exists a positive constant C such that*

$$\|(-\Delta)^{-\sigma/2} \varphi\|_{L^{p_*}(\mathbb{R}^n)} \leq C \|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$.

Lemma 2.3 (Gagliardo-Nirenberg inequality [9, 19, 27]). *Let $n \geq 1$, $0 < \sigma < s < n$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = (1 - \frac{\sigma}{s}) \frac{1}{p_1} + \frac{\sigma}{s} \frac{1}{p_2}$. Then*

$$\|(-\Delta)^{\sigma/2} \varphi\|_{L^p(\mathbb{R}^n)} \leq C \|\varphi\|_{L^{p_1}(\mathbb{R}^n)}^{1-\frac{\sigma}{s}} \|(-\Delta)^{s/2} \varphi\|_{L^{p_2}(\mathbb{R}^n)}^{\frac{\sigma}{s}}$$

holds.

The following estimate is due to [22].

Lemma 2.4 (Kato-Ponce's commutator estimates [17, 22]). *Let $s > 0$ and $1 < p < \infty$. Then*

$$\| [(-\Delta)^{s/2}, g] f \|_{L^p(\mathbb{R}^n)} \leq C (\|\nabla g\|_{L^{p_1}(\mathbb{R}^n)} \|(-\Delta)^{(s-1)/2} f\|_{L^{p_2}(\mathbb{R}^n)} + \|(-\Delta)^{s/2} g\|_{L^{p_3}(\mathbb{R}^n)} \|f\|_{L^{p_4}(\mathbb{R}^n)})$$

and

$$\|(-\Delta)^{s/2}(fg)\|_{L^p(\mathbb{R}^n)} \leq C (\|f\|_{L^{p_1}(\mathbb{R}^n)} \|(-\Delta)^{s/2} g\|_{L^{p_2}(\mathbb{R}^n)} + \|(-\Delta)^{s/2} f\|_{L^{p_3}(\mathbb{R}^n)} \|g\|_{L^{p_4}(\mathbb{R}^n)})$$

with $1 < p_j \leq \infty$ ($j = 1, 4$) and $1 < p_j < \infty$ ($j = 2, 3$) such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

The Hörmander-Mikhlin type inequality (cf. [34, Theorem 3.1]) yields that

$$|\partial_t^m \nabla^\alpha G_\theta(1, x)| \leq C(1 + |x|)^{-n-\theta-\theta m-|\alpha|}$$

for $m \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$. A coupling of this and the scaling property

$$\partial_t^m \nabla^\alpha G_\theta(t, x) = t^{-\frac{n}{\theta}-m-\frac{|\alpha|}{\theta}} \partial_t^m \nabla^\alpha G_\theta(1, x)$$

provides the following lemmas.

Lemma 2.5. *Let $n \geq 1$, $\theta > 0$, $m \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^n$ and $1 \leq p \leq q \leq \infty$. Then there exists a positive constant C such that*

$$\|\partial_t^m \nabla^\alpha G_\theta(t) * \varphi\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{\theta}(\frac{1}{p}-\frac{1}{q})-m-\frac{|\alpha|}{\theta}} \|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$.

Lemma 2.6. Let $N \in \mathbb{Z}_+$, $\varphi \in L^1(\mathbb{R}^n, (1 + |x|)^N dx)$ and $1 \leq q \leq \infty$. Then

$$\left\| G_\theta(t) * \varphi - \sum_{|\alpha| \leq N} \frac{\nabla^\alpha G_\theta(t)}{\alpha!} \int_{\mathbb{R}^n} (-y)^\alpha \varphi(y) dy \right\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{N}{\theta}})$$

as $t \rightarrow \infty$. In addition, if $\varphi \in L^1(\mathbb{R}^n, (1 + |x|^2)^{(N+1)/2} dx)$, then

$$\left\| |x|^\mu \left(G_\theta(t) * \varphi - \sum_{|\alpha| \leq N} \frac{\nabla^\alpha G_\theta(t)}{\alpha!} \int_{\mathbb{R}^n} (-y)^\alpha \varphi(y) dy \right) \right\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{N+1}{\theta}+\frac{\mu}{\theta}}$$

for $0 \leq \mu \leq N$ and $t > 0$.

The solution of (1.1) satisfies the following estimate.

Proposition 2.7. Let $n \geq 2$, $0 < \theta \leq 1$ and $\sigma \geq 0$. Assume that the solution u satisfies (1.2) and (1.3). Then there exist positive constants C and T such that

$$\|(-\Delta)^{\sigma/2} u(t)\|_{L^2(\mathbb{R}^n)} \leq C t^{-\frac{n}{2\theta}-\frac{\sigma}{\theta}}$$

for any $t \geq T$.

Proof. Let $q > \frac{n}{\theta} + \frac{2\sigma}{\theta}$. Using (1.1), we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(t^q \|(-\Delta)^{\sigma/2} u(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + t^q \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}} u(t)\|_{L^2(\mathbb{R}^n)}^2 \\ &= t^q \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u \nabla(-\Delta)^{\sigma/2} \cdot (u \nabla \psi) dx + \frac{q}{2} t^{q-1} \|(-\Delta)^{\sigma/2} u(t)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Since

$$\begin{aligned} & \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u \nabla(-\Delta)^{\sigma/2} \cdot (u \nabla \psi) dx \\ &= \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u \nabla(-\Delta)^{\sigma/2} u \cdot \nabla \psi dx + \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u [\nabla(-\Delta)^{\sigma/2}, \nabla \psi] u dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n} u |(-\Delta)^{\sigma/2} u|^2 dx + \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u [\nabla(-\Delta)^{\sigma/2}, \nabla \psi] u dx, \end{aligned}$$

we have that

$$\begin{aligned} (2.1) \quad & \frac{1}{2} \frac{d}{dt} \left(t^q \|(-\Delta)^{\sigma/2} u(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + t^q \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}} u(t)\|_{L^2(\mathbb{R}^n)}^2 \\ &= \frac{1}{2} t^q \int_{\mathbb{R}^n} u |(-\Delta)^{\sigma/2} u|^2 dx + t^q \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u [\nabla(-\Delta)^{\sigma/2}, \nabla \psi] u dx \\ & \quad + \frac{q}{2} t^{q-1} \|(-\Delta)^{\sigma/2} u(t)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Let $\frac{1}{\rho} = \frac{1}{2} - \frac{\theta}{2n}$, then, from (1.2), we see that

$$\begin{aligned} \int_{\mathbb{R}^n} u |(-\Delta)^{\sigma/2} u|^2 dx &\leq \|u\|_{L^{\rho/(\rho-2)}(\mathbb{R}^n)} \|(-\Delta)^{\sigma/2} u\|_{L^\rho(\mathbb{R}^n)}^2 \leq C t^{-\frac{n}{\theta}(1-\frac{\theta}{n})} \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}} u\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \frac{1}{4} \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}} u\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

for sufficiently large t . The Hölder inequality yields that

$$\int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u [\nabla(-\Delta)^{\sigma/2}, \nabla \psi] u dx \leq \|(-\Delta)^{\sigma/2} u\|_{L^\rho(\mathbb{R}^n)} \|[\nabla(-\Delta)^{\sigma/2}, \nabla \psi] u\|_{L^{\rho'}(\mathbb{R}^n)},$$

where $\frac{1}{\rho} = \frac{1}{2} - \frac{\theta}{2n}$ and $\frac{1}{\rho'} = \frac{1}{2} + \frac{\theta}{2n}$. Using Lemma 2.4 and (1.2), we see that

$$\begin{aligned} & \left\| [\nabla(-\Delta)^{\sigma/2}, \nabla\psi]u \right\|_{L^{\rho'}(\mathbb{R}^n)} \\ & \leq C \left(\|(-\Delta)^{\frac{\sigma+1}{2}}\nabla\psi\|_{L^\rho(\mathbb{R}^n)} \|u\|_{L^{n/\theta}(\mathbb{R}^n)} + \|\nabla^2\psi\|_{L^{n/\theta}(\mathbb{R}^n)} \|(-\Delta)^{\sigma/2}u\|_{L^\rho(\mathbb{R}^n)} \right) \\ & \leq C \|u\|_{L^{n/\theta}(\mathbb{R}^n)} \|(-\Delta)^{\sigma/2}u\|_{L^\rho(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}+1} \|(-\Delta)^{\sigma/2}u\|_{L^\rho(\mathbb{R}^n)}. \end{aligned}$$

The Sobolev inequality says that

$$\|(-\Delta)^{\sigma/2}u\|_{L^\rho(\mathbb{R}^n)} \leq C \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u\|_{L^2(\mathbb{R}^n)}.$$

Thus we have that

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^{\sigma/2}u [\nabla(-\Delta)^{\sigma/2}, \nabla\psi]u dx & \leq C(1+t)^{-\frac{n}{\theta}+1} \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \frac{1}{8} \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

for sufficiently large t . The third term on the right-hand side of (2.1) is treated by Lemma 2.3. Namely, for $\lambda = \frac{2\sigma}{2\sigma+\theta}$, we see that

$$\begin{aligned} t^{q-1} \|(-\Delta)^{\sigma/2}u(t)\|_{L^2(\mathbb{R}^n)}^2 & \leq t^{q-1} \|u(t)\|_{L^2(\mathbb{R}^n)}^{2(1-\lambda)} \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u(t)\|_{L^2(\mathbb{R}^n)}^{2\lambda} \\ & \leq C t^{q-1-\frac{n}{\theta}-\frac{2\sigma}{\theta}} \|u(t)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{8} t^q \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u(t)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Therefore we obtain that

$$\frac{d}{dt} \left(t^q \|(-\Delta)^{\sigma/2}u(t)\|_{L^2(\mathbb{R}^n)}^2 \right) + t^q \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u(t)\|_{L^2(\mathbb{R}^n)}^2 \leq C t^{q-1-\frac{n}{\theta}-\frac{2\sigma}{\theta}}$$

for large t . If we choose sufficiently large T , then, we conclude that

$$\begin{aligned} & t^q \|(-\Delta)^{\sigma/2}u(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_T^t s^q \|(-\Delta)^{\frac{\sigma}{2}+\frac{\theta}{4}}u(s)\|_{L^2(\mathbb{R}^n)}^2 ds \\ & \leq T^q \|(-\Delta)^{\sigma/2}u(T)\|_{L^2(\mathbb{R}^n)}^2 + C \int_T^t s^{q-1-\frac{n}{\theta}-\frac{2\sigma}{\theta}} ds \end{aligned}$$

for $t \geq T$, and complete the proof. \square

The decay of the conservation force field $\nabla\psi$ is given in the following.

Proposition 2.8. *Upon (1.2), $\nabla\psi = \nabla(-\Delta)^{-1}u$ on (1.1) fulfills (1.4) for $\frac{n}{n-1} < p \leq \infty$.*

Proof. Lemma 2.2 and (1.2) give the assertion for $\frac{n}{n-1} < p < \infty$. Since

$$\nabla(-\Delta)^{-1}\varphi(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \varphi(y) dy,$$

we see

$$\begin{aligned} |\nabla(-\Delta)^{-1}u(t)| & \leq C \left(\int_{|x-y| \leq (1+t)^{1/\theta}} + \int_{|x-y| \geq (1+t)^{1/\theta}} \right) \frac{|u(t,y)|}{|x-y|^{n-1}} dy \\ & \leq C \left((1+t)^{\frac{1}{\theta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + (1+t)^{-\frac{n}{\theta}+\frac{1}{\theta}} \|u(t)\|_{L^1(\mathbb{R}^n)} \right). \end{aligned}$$

This inequality together with (1.2) leads the assertion for $p = \infty$. \square

The moment of the solution fulfills the following estimate.

Proposition 2.9. *Let $n \geq 2$, $0 < \theta \leq 1$ and the solution u of (1.1) satisfy (1.2). Assume that $xu_0 \in L^{n/(n-1)}(\mathbb{R}^n)$. Then*

$$\|x_j u(t)\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C \log(e+t)$$

for $j = 1, \dots, n$.

Proof. Let $p = \frac{n}{n-1}$. Multiplying the first equation in (1.1) by $x_j |x_j u|^{p-2} x_j u$ and integrate over \mathbb{R}^n , we have that

$$(2.2) \quad \begin{aligned} & \frac{1}{p} \frac{d}{dt} \|x_j u\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} |x_j u|^{p-2} x_j u (-\Delta)^{\theta/2} (x_j u) dx \\ &= - \int_{\mathbb{R}^n} |x_j u|^{p-2} x_j u [x_j, (-\Delta)^{\theta/2}] u dx + \int_{\mathbb{R}^n} x_j |x_j u|^{p-2} x_j u \nabla u \cdot \nabla (-\Delta)^{-1} u dx - \int_{\mathbb{R}^n} |x_j u|^p u dx. \end{aligned}$$

Lemma 2.1 implies the positivity of the second term in the left hand side of the above equality. The relation $[x_j, (-\Delta)^{\theta/2}] = \theta (-\Delta)^{\frac{\theta-2}{2}} \partial_j$, the Hölder inequality, and Hardy-Littlewood-Sobolev's inequality together with (1.2) provide that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |x_j u|^{p-2} x_j u [x_j, (-\Delta)^{\theta/2}] u dx \right| &\leq C \|(-\Delta)^{\frac{\theta-2}{2}} \partial_j u\|_{L^p(\mathbb{R}^n)} \|x_j u\|_{L^p(\mathbb{R}^n)}^{p-1} \\ &\leq C(1+t)^{-1} \|x_j u\|_{L^p(\mathbb{R}^n)}^{p-1}. \end{aligned}$$

Similarly we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^n} x_j |x_j u|^{p-2} x_j u \nabla u \cdot \nabla (-\Delta)^{-1} u dx - \int_{\mathbb{R}^n} |x_j u|^p u dx \\ &= \int_{\mathbb{R}^n} |x_j u|^{p-2} x_j u \nabla (x_j u) \cdot \nabla (-\Delta)^{-1} u dx - \int_{\mathbb{R}^n} |x_j u|^p u dx + \int_{\mathbb{R}^n} |x_j u|^{p-2} x_j u [x_j, \nabla] u \cdot \nabla (-\Delta)^{-1} u dx \\ &= \left(\frac{1}{p} - 1\right) \int_{\mathbb{R}^n} |x_j u|^p u dx + \int_{\mathbb{R}^n} u \partial_j (-\Delta)^{-1} u |x_j u|^{p-2} x_j u dx. \end{aligned}$$

The Hölder inequality, the Sobolev inequality and (1.2) yield that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} x_j |x_j u|^{p-2} x_j u \nabla u \cdot \nabla (-\Delta)^{-1} u dx - \int_{\mathbb{R}^n} |x_j u|^p u dx \right| \\ &\leq C (\|u\|_{L^\infty(\mathbb{R}^n)} \|x_j u\|_{L^p(\mathbb{R}^n)}^p + \|u \partial_j (-\Delta)^{-1} u\|_{L^p(\mathbb{R}^n)} \|x_j u\|_{L^p(\mathbb{R}^n)}^{p-1}) \\ &\leq C(1+t)^{-n/\theta} (1 + \|x_j u\|_{L^p(\mathbb{R}^n)}) \|x_j u\|_{L^p(\mathbb{R}^n)}^{p-1} \\ &\leq C \left((1+t)^{-1} \|x_j u\|_{L^p(\mathbb{R}^n)}^{p-1} + (1+t)^{-n/\theta} \|x_j u\|_{L^p(\mathbb{R}^n)}^p \right). \end{aligned}$$

Therefore, from (2.2), we obtain the relation

$$f'(t) \leq C_0 \left((1+t)^{-1} f(t)^{1/n} + (1+t)^{-n/\theta} f(t) \right)$$

for $f(t) = \|x_j u(t)\|_{L^p(\mathbb{R}^n)}^p$. Let $g(t) = \exp(-C_0 \int_0^t (1+s)^{-n/\theta} ds)$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon \leq g(t) \leq \varepsilon^{-1}$ for any t , and we see that

$$(f(t)g(t))' \leq C_0(1+t)^{-1} f(t)^{1/n} g(t) \leq C(1+t)^{-1} (f(t)g(t))^{1/n}.$$

Solving this inequality, we complete the proof. \square

Since $L^1(\mathbb{R}^n, (1+|x|)^2 dx) \cap L^\infty(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n, (1+|x|) dx)$, the assertion of Proposition 2.9 is satisfied upon the assumption of our main theorems. Before closing this section, we show the asymptotic profile of the solution.

Proposition 2.10. *Let $n \geq 2$, $0 < \theta \leq 1$, $1 \leq q \leq \infty$, $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and the solution u of (1.1) fulfill (1.2) and (1.3). Then*

$$\|u(t) - MG_\theta(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{p})})$$

as $t \rightarrow \infty$, where $M = \int_{\mathbb{R}^n} u_0(y) dy$. In addition, if $xu_0 \in L^1(\mathbb{R}^n)$, then

$$\|u(t) - MG_\theta(t)\|_{L^q(\mathbb{R}^n)} \leq \begin{cases} Ct^{-\frac{n}{\theta}(1-\frac{1}{q})} (1+t)^{-\frac{1}{\theta}} & (n \geq 3 \text{ or } \theta < 1) \\ Ct^{-2(1-\frac{1}{q})} (1+t)^{-1} \log(e+t) & (n = 2 \text{ and } \theta = 1) \end{cases}$$

for $t > 0$.

Proof. By (1.5), we see that

$$(2.3) \quad u(t) - MG_\theta(t) = G_\theta(t) * u_0 - MG_\theta(t) + \int_0^t \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds.$$

Since the estimate for the linear part is well-known, we consider the nonlinear term. By Lemmas 2.5 and 2.2, and (1.2), we have that

$$\begin{aligned} & \left\| \int_0^{t/2} \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^q(\mathbb{R}^n)} \\ & \leq C \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{1}{\theta}} \|u \nabla(-\Delta)^{-1}u\|_{L^1(\mathbb{R}^n)} ds \\ & \leq C \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{1}{\theta}} (1+s)^{-\frac{n-1}{\theta}} ds. \end{aligned}$$

Thus

$$(2.4) \quad \begin{aligned} & \left\| \int_0^{t/2} \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^q(\mathbb{R}^n)} \\ & \leq \begin{cases} Ct^{-\frac{n}{\theta}(1-\frac{1}{q})}(1+t)^{-\frac{1}{\theta}} & (n \geq 3 \text{ or } \theta < 1) \\ Ct^{-2(1-\frac{1}{q})}(1+t)^{-1} \log(e+t) & (n = 2 \text{ and } \theta = 1). \end{cases} \end{aligned}$$

When $n \geq 3$, we obtain by Lemmas 2.5 and 2.2, Proposition 2.7, and (1.2) that

$$\begin{aligned} & \left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^1(\mathbb{R}^n)} \\ & \leq C \int_{t/2}^t \left(\|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1}u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}^2 \right) ds \\ & \leq C \int_{t/2}^t (1+s)^{-\frac{n}{\theta}} ds. \end{aligned}$$

When $n = 2$, Lemma 2.5 gives that

$$\begin{aligned} & \left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^1(\mathbb{R}^2)} \\ & \leq C \int_{t/2}^t \left(\|\nabla u\|_{L^{9/5}(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}u\|_{L^{9/4}(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)}^2 \right) ds. \end{aligned}$$

Here the Gagliardo-Nirenberg inequality yields that

$$\|\nabla u\|_{L^{9/5}(\mathbb{R}^2)} \leq C \|u\|_{L^{3/2}(\mathbb{R}^2)}^{1/3} \|(-\Delta)^{3/4}u\|_{L^2(\mathbb{R}^2)}^{2/3}.$$

Therefore Lemmas 2.2 and 2.7, and (1.2) conclude that

$$\left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^1(\mathbb{R}^2)} \leq C \int_{t/2}^t (1+s)^{-\frac{2}{\theta}} ds.$$

If we put $\sigma = \frac{(1-\varepsilon)n}{2}$ and $\frac{1}{r} = \frac{\varepsilon}{2}$ for some small $\varepsilon > 0$, then, by Lemma 2.5 and the Sobolev inequality, we obtain that

$$\begin{aligned} & \left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \\ & \leq C \int_{t/2}^t (t-s)^{-\frac{n}{\theta r}} (\|\nabla u\|_{L^r(\mathbb{R}^n)} \|\nabla (-\Delta)^{-1} u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^{2r}(\mathbb{R}^n)}^2) ds \\ & \leq \int_{t/2}^t (t-s)^{-\frac{\varepsilon n}{2\theta}} (\|\nabla (-\Delta)^{\sigma/2} u\|_{L^2(\mathbb{R}^n)} \|\nabla (-\Delta)^{-1} u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^{2r}(\mathbb{R}^n)}^2) ds. \end{aligned}$$

Hence, by Propositions 2.7 and 2.8, and (1.2), we obtain that

$$\left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds \right\|_{L^\infty(\mathbb{R}^n)} \leq C \int_{t/2}^t (t-s)^{-\frac{\varepsilon n}{2\theta}} (1+s)^{-\frac{2n}{\theta} + \frac{\varepsilon n}{2\theta}} ds.$$

Thus, by the Hölder inequality, we conclude that

$$(2.5) \quad \left\| \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds \right\|_{L^q(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{q}) - \frac{n}{\theta} + 1}$$

for $1 \leq q \leq \infty$. Applying (2.4) and (2.5) to (2.3), we complete the proof. \square

3. PROOF OF MAIN RESULTS

3.1. Proof of Theorem 1.1. In (1.5), large-time behavior of $G_\theta(t) * u_0$ is well-known. We split the nonlinear term into

$$(3.1) \quad \begin{aligned} & \int_0^t \nabla G_\theta(t-s) * (u \nabla (-\Delta)^{-1} u)(s) ds \\ & = \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t) \cdot \int_0^\infty \int_{\mathbb{R}^n} (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds + r_1(t) + r_2(t) + r_3(t), \end{aligned}$$

where

$$\begin{aligned} r_1(t) &= \int_0^{t/2} \int_{\mathbb{R}^n} (\nabla G_\theta(t-s, x-y) - \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t, x) (-y)^\beta) \cdot (u \nabla (-\Delta)^{-1} u)(s, y) dy ds, \\ r_2(t) &= \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds, \\ r_3(t) &= - \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t) \cdot \int_{t/2}^\infty \int_{\mathbb{R}^n} (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds. \end{aligned}$$

Since $\int_{\mathbb{R}^n} u \nabla (-\Delta)^{-1} u dy = 0$, r_1 is represented by

$$\begin{aligned} r_1(t) &= \int_0^{t/2} \int_{\mathbb{R}^n} (\nabla G_\theta(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla G_\theta(t-s, x) (-y)^\beta) \cdot (u \nabla (-\Delta)^{-1} u)(s, y) dy ds \\ &+ \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^n} (\nabla^\beta \nabla G_\theta(t-s, x) - \nabla^\beta \nabla G_\theta(t, x)) \cdot (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds. \end{aligned}$$

For some $R(t) = o(t^{1/\theta})$ ($t \rightarrow \infty$), we divide r_1 to $r_1 = r_{1,1} + r_{1,2} + r_{1,3}$, where

$$\begin{aligned} r_{1,1}(t) &= \int_0^{t/2} \int_{|y| \leq R(t)} \left(\nabla G_\theta(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla G_\theta(t-s, x) (-y)^\beta \right) \cdot (u \nabla (-\Delta)^{-1} u)(s, y) dy ds, \\ r_{1,2}(t) &= \int_0^{t/2} \int_{|y| > R(t)} \left(\nabla G_\theta(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla G_\theta(t-s, x) (-y)^\beta \right) \cdot (u \nabla (-\Delta)^{-1} u)(s, y) dy ds, \\ r_{1,3}(t) &= \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^n} \left(\nabla^\beta \nabla G_\theta(t-s, x) - \nabla^\beta \nabla G_\theta(t, x) \right) \cdot (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds. \end{aligned}$$

Taylor's theorem yields that

$$\begin{aligned} r_{1,1}(t) &= \sum_{|\beta|=2} \int_0^{t/2} \int_{|y| \leq R(t)} \int_0^1 \frac{\nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y)}{\beta!} \cdot \lambda (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) d\lambda dy ds, \\ r_{1,2}(t) &= \sum_{|\beta|=1} \int_0^{t/2} \int_{|y| > R(t)} \left(\int_0^1 \nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y) d\lambda + \nabla^\beta \nabla G_\theta(t-s, x) \right) \\ &\quad \cdot (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s, y) dy ds, \\ r_{1,3}(t) &= \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^n} \int_0^1 \partial_t \nabla^\beta \nabla G_\theta(t-s+\lambda s, x) \cdot (-s) (-y)^\beta (u \nabla (-\Delta)^{-1} u)(s) d\lambda dy ds. \end{aligned}$$

By Lemma 2.5 and Propositions 2.8 and 2.9, we have that

$$\begin{aligned} \|r_{1,1}(t)\|_{L^q(\mathbb{R}^n)} &\leq CR(t) \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{3}{\theta}} \|y(u \nabla (-\Delta)^{-1} u)(s)\|_{L^1(\mathbb{R}^n)} ds \\ &\leq CR(t) \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{3}{\theta}} (1+s)^{-\frac{n-2}{\theta}} \log(e+s) ds. \end{aligned}$$

Thus

$$\|r_{1,1}(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. Similarly

$$\|r_{1,2}(t)\|_{L^q(\mathbb{R}^n)} \leq C \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} \|y(u \nabla (-\Delta)^{-1} u)(s, y)\|_{L^1(|y| \geq R(t))} ds.$$

Hence, by Lebesgue's monotone convergence theorem together with

$$\begin{aligned} &\int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} \|y(u \nabla (-\Delta)^{-1} u)(s, y)\|_{L^1(\mathbb{R}^n)} ds \\ &\leq C \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} (1+s)^{-\frac{n-2}{\theta}} \log(e+s) ds = O(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}}), \end{aligned}$$

we conclude that

$$\|r_{1,2}(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. Moreover

$$\|r_{1,3}(t)\|_{L^q(\mathbb{R}^n)} \leq C \int_0^{t/2} (t-s)^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}-1} (1+s)^{-\frac{n-2}{\theta}} \log(e+s) ds.$$

Thus

$$\|r_{1,3}(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. Consequently

$$(3.2) \quad \|r_1(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. The inequality (2.5) leads that

$$(3.3) \quad \|r_2(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. Propositions 2.8 and 2.9 provide that

$$\|r_3(t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} \int_{t/2}^{\infty} s^{-\frac{n-2}{\theta}} \log(e+s) ds$$

and

$$(3.4) \quad \|r_3(t)\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. Applying (3.2), (3.3) and (3.4) to (3.1), we complete the proof. \square

3.2. Proof of Theorem 1.2. To show Theorems 1.2 and 1.3, we prepare the following estimates.

Proposition 3.1. *Let $n \geq 2$, $0 < \theta \leq 1$ and $\sigma > 0$. Assume that the solution u of (1.1) satisfies (1.2) and (1.3). Then there exist positive constants C and T such that*

$$\|(-\Delta)^{\sigma/2} (u(t) - MG_{\theta}(t))\|_{L^2(\mathbb{R}^n)} \leq \begin{cases} Ct^{-\frac{n}{2\theta}-\frac{\sigma}{\theta}}(1+t)^{-\frac{1}{\theta}} & (n \geq 3 \text{ or } \theta < 1) \\ Ct^{-1-\sigma}(1+t)^{-1} \log(e+t) & (n = 2 \text{ and } \theta = 1) \end{cases}$$

for $t \geq T$, where $M = \int_{\mathbb{R}^n} u_0(y) dy$.

Proof. We consider only the nonlinear term of (1.5). By Lemma 2.5, we see that

$$\begin{aligned} & \left\| (-\Delta)^{\sigma/2} \int_0^t \nabla G_{\theta}(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds \right\|_{L^2(\mathbb{R}^n)} \\ & \leq \left\| \int_0^{t/2} \nabla(-\Delta)^{\sigma/2} G_{\theta}(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds \right\|_{L^2(\mathbb{R}^n)} \\ & \quad + \left\| \int_{t/2}^t G_{\theta}(t-s) * \nabla(-\Delta)^{\sigma/2} \cdot (u \nabla(-\Delta)^{-1} u)(s) ds \right\|_{L^2(\mathbb{R}^n)} \\ & \leq C \int_0^{t/2} (t-s)^{-\frac{n}{2\theta}-\frac{1+\sigma}{\theta}} \|(u \nabla(-\Delta)^{-1} u)(s)\|_{L^1(\mathbb{R}^n)} ds \\ & \quad + C \int_{t/2}^t \|\nabla(-\Delta)^{\sigma/2} \cdot (u \nabla(-\Delta)^{-1} u)(s)\|_{L^2(\mathbb{R}^n)} ds. \end{aligned}$$

The Hölder inequality, Proposition 2.8 and (1.2) yield that

$$\|(u \nabla(-\Delta)^{-1} u)(s)\|_{L^1(\mathbb{R}^n)} \leq C(1+s)^{-\frac{n-1}{\theta}}$$

for $s > 0$. From Lemma 2.4, we have that

$$\begin{aligned} & \|\nabla(-\Delta)^{\sigma/2} \cdot (u \nabla(-\Delta)^{-1} u)(s)\|_{L^2(\mathbb{R}^n)} \\ & \leq C \left(\|\nabla(-\Delta)^{\sigma/2} u\|_{L^4(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1} u\|_{L^4(\mathbb{R}^n)} + \|u\|_{L^4(\mathbb{R}^n)} \|\nabla^2(-\Delta)^{\frac{\sigma}{2}-1} u\|_{L^4(\mathbb{R}^n)} \right). \end{aligned}$$

Hence, by a coupling of the Sobolev inequality and Proposition 2.7, Proposition 2.8, and (1.2), we obtain that

$$\|\nabla(-\Delta)^{\sigma/2} \cdot (u \nabla(-\Delta)^{-1} u)(s)\|_{L^2(\mathbb{R}^n)} \leq Cs^{-\frac{n}{2\theta}-\frac{n}{\theta}-\frac{\sigma}{\theta}}$$

for large s . Therefore we complete the proof. \square

Proposition 3.2. *Let $n \geq 2$, $0 < \theta \leq 1$ and $\varepsilon > 0$. Assume that $u_0 \in L^1(\mathbb{R}^n, (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^n)$, and the solution u of (1.1) satisfies (1.2). Then there exists positive constant C such that*

$$\|\nabla(-\Delta)^{-1} (u(t) - MG_{\theta}(t))\|_{L^2(\mathbb{R}^n)} \leq \begin{cases} Ct^{-\frac{\varepsilon}{\theta}}(1+t)^{-\frac{n}{2\theta}+\frac{\varepsilon}{\theta}} & (n \geq 3 \text{ or } \theta < 1) \\ Ct^{-\varepsilon}(1+t)^{-1+\varepsilon} \log(e+t) & (n = 2 \text{ and } \theta = 1) \end{cases}$$

for $t > 0$, where $M = \int_{\mathbb{R}^n} u_0(y) dy$.

Proof. For $k = 1, \dots, n$, we see from (1.5) that

$$\begin{aligned} & \partial_k(-\Delta)^{-1}(u - MG_\theta) \\ &= \partial_k(-\Delta)^{-1}(G_\theta(t) * u_0 - MG_\theta) + \partial_k(-\Delta)^{-1} \int_0^t G_\theta(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1}u)(s) ds \\ &= \int_{\mathbb{R}^n} \int_0^1 \partial_k(-\Delta)^{-1} \nabla G_\theta(t, x-y+\lambda y) \cdot (-y) u_0(y) d\lambda dy \\ & \quad + \int_0^t \partial_k(-\Delta)^{-1} \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds. \end{aligned}$$

Here we used Taylor's theorem. By Lemma 2.5, we see that

$$\left\| \int_{\mathbb{R}^2} \int_0^1 \partial_k(-\Delta)^{-1} \nabla G_\theta(t, x-y+\lambda y) \cdot (-y) u_0(y) d\lambda dy \right\|_{L^2(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\theta}} \|yu_0\|_{L^1(\mathbb{R}^n)}.$$

Since $u_0 \in L^1(\mathbb{R}^n, (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^n) \subset L^r(\mathbb{R}^n, (1+|x|)dx)$ for $\frac{1}{r} = \frac{1}{2} + \frac{\varepsilon}{n}$, we obtain that

$$\left\| \int_{\mathbb{R}^2} \int_0^1 \partial_k(-\Delta)^{-1} \nabla G_\theta(t, x-y+\lambda y) \cdot (-y) u_0(y) d\lambda dy \right\|_{L^2(\mathbb{R}^n)} \leq Ct^{-\frac{\varepsilon}{\theta}} \|yu_0\|_{L^r(\mathbb{R}^n)}.$$

Thus

$$\left\| \int_{\mathbb{R}^2} \int_0^1 \partial_k(-\Delta)^{-1} \nabla G_\theta(t, x-y+\lambda y) \cdot (-y) u_0(y) d\lambda dy \right\|_{L^2(\mathbb{R}^n)} \leq Ct^{-\frac{\varepsilon}{\theta}} (1+t)^{-\frac{n}{2\theta} + \frac{\varepsilon}{\theta}}.$$

Similarly, we obtain that

$$\begin{aligned} & \left\| \int_0^t \partial_k(-\Delta)^{-1} \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^2(\mathbb{R}^n)} \\ & \leq C \int_0^{t/2} (t-s)^{-\frac{n}{2\theta}} \|u \nabla(-\Delta)^{-1}u(s)\|_{L^1(\mathbb{R}^n)} ds + C \int_{t/2}^t (t-s)^{-\frac{\varepsilon}{\theta}} \|u \nabla(-\Delta)^{-1}u(s)\|_{L^r(\mathbb{R}^n)} ds \\ & \leq C \int_0^{t/2} (t-s)^{-\frac{n}{2\theta}} (1+s)^{-\frac{n-1}{\theta}} ds + C \int_{t/2}^t (t-s)^{-\frac{\varepsilon}{\theta}} (1+s)^{-\frac{n}{2\theta} - \frac{n-1}{\theta} + \frac{\varepsilon}{\theta}} ds \\ & \leq \begin{cases} Ct^{-\frac{n}{2\theta}} & (n \geq 3 \text{ or } \theta < 1) \\ Ct^{-1} \log(e+t) & (n = 2 \text{ and } \theta = 1) \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^t \partial_k(-\Delta)^{-1} \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1}u)(s) ds \right\|_{L^2(\mathbb{R}^n)} \\ & \leq C \int_0^t (t-s)^{-\frac{\varepsilon}{\theta}} \|u \nabla(-\Delta)^{-1}u(s)\|_{L^r(\mathbb{R}^n)} ds \leq C. \end{aligned}$$

Therefore we complete the proof. \square

Lemma 2.2, and Propositions 2.9, 2.10 and 3.2 affirm (1.8) when $\theta < n-1$.

Proof of Theorem 1.2. We split the nonlinear term on (1.5) as follows:

$$\begin{aligned}
(3.5) \quad & \int_0^t \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds \\
&= M^2 J(t) + \int_0^t \nabla G_\theta(t-s) * (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s) ds \\
&= M^2 J(t) + \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t, x) \cdot \int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds \\
&\quad + \tilde{r}_1(t) + \tilde{r}_2(t) + \tilde{r}_3(t),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{r}_1(t) &= \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla G_\theta(t-s, x-y) - \sum_{|\beta|=1} \nabla G_\theta(t, x) (-y)^\beta) \\
&\quad \cdot (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds, \\
\tilde{r}_2(t) &= \int_{t/2}^t G_\theta(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s) ds, \\
\tilde{r}_3(t) &= - \sum_{|\beta|=1} \nabla^\beta \nabla G_\theta(t) \cdot \int_{t/2}^\infty \int_{\mathbb{R}^2} (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds.
\end{aligned}$$

Here we used the relation $\int_{\mathbb{R}^2} (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy = 0$ for \tilde{r}_1 . For some $R(t) > 0$, $R(t) = o(t^{1/\theta})$ as $t \rightarrow \infty$, by the similar argument as in the proof of Theorem 1.1, we divide \tilde{r}_1 into

$$\begin{aligned}
(3.6) \quad \tilde{r}_1(t) &= \sum_{|\beta|=2} \int_0^{t/2} \int_{|y| \leq R(t)} \int_0^1 \frac{\nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y)}{\beta!} \\
&\quad \cdot (-\lambda)(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) d\lambda dy ds \\
&\quad + \sum_{|\beta|=1} \int_0^{t/2} \int_{|y| > R(t)} \left(\int_0^1 \nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y) d\lambda + \nabla^\beta \nabla G_\theta(t-s, x) \right) \\
&\quad \cdot (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds \\
&\quad + \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \partial_t \nabla^\beta \nabla G_\theta(t-s+\lambda s, x) \\
&\quad \cdot (-s)(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) d\lambda dy ds.
\end{aligned}$$

Since $u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta = u \nabla(-\Delta)^{-1} (u - M G_\theta) + M(u - M G_\theta) \nabla(-\Delta)^{-1} G_\theta$, we see from (1.2), and Propositions 2.10 and 3.2 that

$$\begin{aligned}
& \|y_j (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)\|_{L^1(\mathbb{R}^2)} \\
& \leq \|y_j u\|_{L^2(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1} (u - M G_\theta)\|_{L^2(\mathbb{R}^2)} + \|u - M G_\theta\|_{L^1(\mathbb{R}^2)} \|y_j \nabla(-\Delta)^{-1} G_\theta\|_{L^\infty(\mathbb{R}^2)} \\
& \leq C s^{-\frac{\varepsilon}{\theta}} (1+s)^{-\frac{1}{\theta} + \frac{\varepsilon}{\theta}} \log(e+s).
\end{aligned}$$

Lemma 2.5 together with the above inequality provides that

$$\begin{aligned}
& \left\| \sum_{|\beta|=2} \int_0^{t/2} \int_{|y| \leq R(t)} \int_0^1 \frac{\nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y)}{\beta!} \right. \\
& \quad \cdot (-\lambda)(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) d\lambda dy ds \Big\|_{L^q(\mathbb{R}^2)} \\
& \leq CR(t) \sum_{|\beta|=1} \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{3}{\theta}} \|(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s)\|_{L^1(\mathbb{R}^2)} ds \\
& \leq CR(t) \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{3}{\theta}} s^{-\frac{\varepsilon}{\theta}} (1+s)^{-\frac{1}{\theta}+\frac{\varepsilon}{\theta}} \log(e+s) ds = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})
\end{aligned}$$

as $t \rightarrow \infty$. In a similar manner to above, we have that

$$\begin{aligned}
& \left\| \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \left(\int_0^1 \nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y) d\lambda + \nabla^\beta \nabla G_\theta(t-s, x) \right) \right. \\
& \quad \cdot (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds \Big\|_{L^q(\mathbb{R}^2)} \\
& \leq C \sum_{|\beta|=1} \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} \|(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s)\|_{L^1(\mathbb{R}^2)} ds \\
& \leq C \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}} s^{-\frac{\varepsilon}{\theta}} (1+s)^{-\frac{1}{\theta}+\frac{\varepsilon}{\theta}} \log(e+s) ds = O(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}}).
\end{aligned}$$

Hence Lebesgue's monotone convergence theorem yields that

$$\begin{aligned}
& \left\| \sum_{|\beta|=1} \int_0^{t/2} \int_{|y| \geq R(t)} \left(\int_0^1 \nabla^\beta \nabla G_\theta(t-s, x-y+\lambda y) d\lambda + \nabla^\beta \nabla G_\theta(t-s, x) \right) \right. \\
& \quad \cdot (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) dy ds \Big\|_{L^q(\mathbb{R}^2)} = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})
\end{aligned}$$

as $t \rightarrow \infty$. Similarly we obtain that

$$\begin{aligned}
& \left\| \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \partial_t \nabla^\beta \nabla G_\theta(t-s+\lambda s, x) \right. \\
& \quad \cdot (-s)(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s, y) d\lambda dy ds \Big\|_{L^q(\mathbb{R}^2)} \\
& \leq C \sum_{|\beta|=1} \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}-1} s \|(-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 G_\theta \nabla(-\Delta)^{-1} G_\theta)(s)\|_{L^1(\mathbb{R}^2)} ds \\
& \leq C \int_0^{t/2} (t-s)^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}-1} s^{1-\frac{\varepsilon}{\theta}} (1+s)^{-\frac{1}{\theta}+\frac{\varepsilon}{\theta}} \log(e+s) ds = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})
\end{aligned}$$

as $t \rightarrow \infty$. Therefore we conclude that

$$(3.7) \quad \|\tilde{r}_1(t)\|_{L^q(\mathbb{R}^2)} = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$. For $1 \leq q < \infty$, we see that

$$\begin{aligned} \|\tilde{r}_2(t)\|_{L^q(\mathbb{R}^2)} \leq & C \int_{t/2}^t \left\{ \|\nabla u\|_{L^{2q}(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}(u - MG_\theta)\|_{L^{2q}(\mathbb{R}^2)} + \|u\|_{L^{2q}(\mathbb{R}^2)} \|u - MG_\theta\|_{L^{2q}(\mathbb{R}^2)} \right. \\ & \left. + \|\nabla(u - MG_\theta)\|_{L^{2q}(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}G_\theta\|_{L^{2q}(\mathbb{R}^2)} + \|u - MG_\theta\|_{L^{2q}(\mathbb{R}^2)} \|G_\theta\|_{L^{2q}(\mathbb{R}^2)} \right\} ds. \end{aligned}$$

From Proposition 3.2, or Proposition 2.10 together with Hardy-Littlewood-Sobolev's inequality leads that

$$\|\nabla(-\Delta)^{-1}(u - MG_\theta)\|_{L^{2q}(\mathbb{R}^2)} \leq Ct^{-\frac{2}{\theta}(1-\frac{1}{2q})} \log(e+t).$$

We choose $\sigma = 1 - \frac{1}{q}$, then, by the Sobolev inequality and Propositions 2.7 and 3.1, we have that

$$\begin{aligned} \|\nabla u\|_{L^{2q}(\mathbb{R}^2)} & \leq C \|\nabla(-\Delta)^{\sigma/2} u\|_{L^2(\mathbb{R}^2)} \leq Ct^{-\frac{2}{\theta}(1-\frac{1}{2q})-\frac{1}{\theta}}, \\ \|\nabla(u - MG_\theta)\|_{L^{2q}(\mathbb{R}^2)} & \leq C \|\nabla(-\Delta)^{\sigma/2}(u - MG_\theta)\|_{L^2(\mathbb{R}^2)} \leq Ct^{-\frac{2}{\theta}(1-\frac{1}{2q})-\frac{2}{\theta}}. \end{aligned}$$

When $q = \infty$, we obtain that

$$\begin{aligned} & \|\tilde{r}_2(t)\|_{L^\infty(\mathbb{R}^2)} \\ \leq & C \int_{t/2}^t (t-s)^{-\frac{2}{\theta p}} \left\{ \|\nabla u\|_{L^{2p}(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}(u - MG_\theta)\|_{L^{2p}(\mathbb{R}^2)} + \|u\|_{L^{2p}(\mathbb{R}^2)} \|u - MG_\theta\|_{L^{2p}(\mathbb{R}^2)} \right. \\ & \left. + \|\nabla(u - MG_\theta)\|_{L^{2p}(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}G_\theta\|_{L^{2p}(\mathbb{R}^2)} + \|u - MG_\theta\|_{L^{2p}(\mathbb{R}^2)} \|G_\theta\|_{L^{2p}(\mathbb{R}^2)} \right\} ds \end{aligned}$$

for some $2/\theta < p < \infty$. Hence we can treat $\|\tilde{r}_2(t)\|_{L^\infty(\mathbb{R}^2)}$ in a similar manner to above. Thus we conclude that

$$(3.8) \quad \|\tilde{r}_2(t)\|_{L^q(\mathbb{R}^2)} \leq \int_{t/2}^t s^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{3}{\theta}} \log(e+s) ds = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Propositions 2.9, 2.10 and 3.2 give that

$$\begin{aligned} & \int_{t/2}^\infty \int_{\mathbb{R}^2} |y_j(u \nabla(-\Delta)^{-1}u - M^2 G_\theta \nabla(-\Delta)^{-1}G_\theta)| dy ds \\ \leq & \int_{t/2}^\infty \left\{ \|y_j u\|_{L^2(\mathbb{R}^2)} \|\nabla(-\Delta)^{-1}(u - MG_\theta)\|_{L^2(\mathbb{R}^2)} + M \|u - MG_\theta\|_{L^1(\mathbb{R}^2)} \|y_j \nabla(-\Delta)^{-1}G_\theta\|_{L^\infty(\mathbb{R}^2)} \right\} ds \\ \leq & C \int_{t/2}^\infty s^{-1/\theta} \log(e+s) ds \end{aligned}$$

and

$$(3.9) \quad \|\tilde{r}_3(t)\|_{L^q(\mathbb{R}^2)} = o(t^{-\frac{2}{\theta}(1-\frac{1}{q})-\frac{2}{\theta}})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Applying (3.7)–(3.9) to (3.5), we complete the proof. \square

3.3. Proof of Theorem 1.3. Lemma 2.6 provides the estimate for the linear term on (1.5). We divide the nonlinear term into

$$\begin{aligned}
& \int_0^t \nabla P(t-s) * (u \nabla (-\Delta)^{-1} u)(s) ds \\
&= M^2 \int_0^t \nabla P(t-s) * (P \nabla (-\Delta)^{-1} P)(1+s) ds \\
& \quad + \int_0^t \nabla P(t-s) * (u \nabla (-\Delta)^{-1} u(s) - M^2 P \nabla (-\Delta)^{-1} P(1+s)) ds \\
&= M^2 \tilde{J}(t) + M^2 \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^{t/2} \int_{\mathbb{R}^3} (-y)^\beta (P \nabla (-\Delta)^{-1} P)(1+s, y) dy ds \\
& \quad + \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^\infty \int_{\mathbb{R}^3} (-y)^\beta (u \nabla (-\Delta)^{-1} u(s, x) - P \nabla (-\Delta)^{-1} P(1+s, y)) dy ds \\
& \quad + \varrho_1(t) + \cdots + \varrho_5(t),
\end{aligned}$$

where

$$\begin{aligned}
\varrho_1(t) &= \int_0^{t/2} \int_{\mathbb{R}^3} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t, x) (-y)^\beta \right) \\
& \quad \cdot (u \nabla (-\Delta)^{-1} u(s, x) - M^2 P \nabla (-\Delta)^{-1} P(1+s, y)) dy ds, \\
\varrho_2(t) &= \int_{t/2}^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u(s, x) - M^2 P \nabla (-\Delta)^{-1} P(1+s, y)) dy ds, \\
\varrho_3(t) &= - \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_{t/2}^\infty \int_{\mathbb{R}^3} (-y)^\beta (u \nabla (-\Delta)^{-1} u(s, x) - M^2 P \nabla (-\Delta)^{-1} P(1+s, y)) dy ds, \\
\varrho_4(t) &= M^2 \int_0^{t/2} \int_{\mathbb{R}^3} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t, x) (-y)^\beta \right) \\
& \quad \cdot (P \nabla (-\Delta)^{-1} P(1+s, y) - P \nabla (-\Delta)^{-1} P(s, y)) dy ds, \\
\varrho_5(t) &= M^2 \int_{t/2}^t P(t-s) * \nabla \cdot (P \nabla (-\Delta)^{-1} P(1+s, y) - P \nabla (-\Delta)^{-1} P(s, y)) dy ds \\
& \quad - M^2 \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_{t/2}^t \int_{\mathbb{R}^3} (-y)^\beta (P \nabla (-\Delta)^{-1} P(1+s, y) - P \nabla (-\Delta)^{-1} P(s, y)) dy ds.
\end{aligned}$$

We note that

$$\begin{aligned}
& \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^{t/2} \int_{\mathbb{R}^3} (-y)^\beta (P \nabla (-\Delta)^{-1} P)(1+s, y) dy ds \\
&= -\frac{1}{3} \Delta P(t) \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}^3} y \cdot (P \nabla (-\Delta)^{-1} P)(1, y) dy = \tilde{K}(t),
\end{aligned}$$

since $P \partial_j (-\Delta)^{-1} P$ is an odd function in x_j . The same argument as in the proof of Theorem 1.2 leads that

$$\|\varrho_1(t)\|_{L^q(\mathbb{R}^3)} + \|\varrho_2(t)\|_{L^q(\mathbb{R}^3)} + \|\varrho_3(t)\|_{L^q(\mathbb{R}^3)} = o(t^{-3(1-\frac{1}{q})-2})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Lemma 2.5 together with Taylor's theorem describes that

$$\begin{aligned} \varrho_4(t) = & \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^3} \int_0^1 \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \lambda(-y)^\beta \\ & \cdot \partial_t (P \nabla(-\Delta)^{-1} P)(s+\mu, y) d\mu d\lambda dy ds \\ & + \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^3} \int_0^1 \int_0^1 \partial_t \nabla^\beta \nabla P(t-s+\lambda s, x)(-s)(-y)^\beta \\ & \cdot \partial_t (P \nabla(-\Delta)^{-1} P)(s+\mu, y) d\mu d\lambda dy ds \end{aligned}$$

and

$$\|\varrho_4(t)\|_{L^q(\mathbb{R}^3)} \leq C \int_0^{t/2} \int_0^1 (t-s)^{-3(1-\frac{1}{q})-3} (s+\mu)^{-1} d\mu ds.$$

Similarly we obtain that

$$\|\varrho_5(t)\|_{L^q(\mathbb{R}^3)} \leq C \int_{t/2}^t \int_0^1 (s+\mu)^{-3(1-\frac{1}{q})-4} d\mu ds + Ct^{-3(1-\frac{1}{q})-2} \int_{t/2}^t \int_0^1 (s+\mu)^{-2} d\mu ds.$$

Therefore ϱ_4 and ϱ_5 fulfill that

$$\|\varrho_4(t)\|_{L^q(\mathbb{R}^3)} + \|\varrho_5(t)\|_{L^q(\mathbb{R}^3)} = o(t^{-3(1-\frac{1}{q})-2})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Therefore we derive the assertion. \square

3.4. Proof of Theorem 1.4. Before proving Theorem 1.4 we prepare the following proposition.

Proposition 3.3. *Upon the assumption of Theorem 1.4,*

$$\begin{aligned} (3.10) \quad & \|u(t) - MP(t) - m \cdot \nabla P(1+t) - M^2 J(1+t)\|_{L^p(\mathbb{R}^2)} \\ & \leq Ct^{-2(1-\frac{1}{p})} (1+t)^{-2} (\log(e+t))^2 \end{aligned}$$

for $t > 0$ and $1 \leq p \leq \infty$. Moreover, for $\sigma > 0$, there exist positive constants C and T such that

$$\begin{aligned} (3.11) \quad & \|(-\Delta)^{\sigma/2} (u(t) - MP(t) - m \cdot \nabla P(1+t) - M^2 J(1+t))\|_{L^2(\mathbb{R}^2)} \\ & \leq Ct^{-1-\sigma} (1+t)^{-2} (\log(e+t))^2 \end{aligned}$$

for $t \geq T$.

Proof. We show (3.11). From (1.5) we see that

$$\begin{aligned} & u(t) - MP(t) - m \cdot \nabla P(1+t) - M^2 J(1+t) \\ & = P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t) + \int_0^t P(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1} u)(s) ds - M^2 J(1+t). \end{aligned}$$

Since

$$\begin{aligned} & P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t) \\ & = P(t) * u_0 - MP(t) - m \cdot \nabla P(t) + m \cdot \nabla (P(t) - P(1+t)) \\ & = \sum_{|\alpha|=2} \int_{\mathbb{R}^2} \int_0^1 \frac{\nabla^\alpha P(t, x-y+\lambda y)}{\alpha!} \lambda(-y)^\alpha u_0(y) d\lambda dy - m \cdot \int_0^1 \partial_t \nabla P(t+\mu) d\mu, \end{aligned}$$

we have for $\sigma > 0$ that

$$\|(-\Delta)^{\sigma/2} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t))\|_{L^2(\mathbb{R}^2)} \leq Ct^{-3-\sigma}.$$

From (1.6), we obtain that

$$\begin{aligned}
& (-\Delta)^{\sigma/2} \left(\int_0^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds - M^2 J(1+t) \right) \\
&= \int_0^{t/2} \nabla (-\Delta)^{\sigma/2} P(t-s) * (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \\
&\quad + \int_{t/2}^t P(t-s) * \nabla (-\Delta)^{\sigma/2} \cdot (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \\
&\quad + M^2 (-\Delta)^{\sigma/2} (J(t) - J(1+t)).
\end{aligned}$$

Taylor's theorem together with the relation $\int_{\mathbb{R}^2} (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P) dy = 0$ gives that

$$\begin{aligned}
& \int_0^{t/2} \nabla (-\Delta)^{\sigma/2} P(t-s) * (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \\
&= \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \nabla^\beta \nabla (-\Delta)^{\sigma/2} P(t-s, x-y+\lambda y) \\
&\quad \cdot (-y)^\beta (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s, y) d\lambda dy ds.
\end{aligned}$$

Hence

$$\begin{aligned}
& \left\| \int_0^{t/2} \nabla (-\Delta)^{\sigma/2} P(t-s) * (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \right\|_{L^2(\mathbb{R}^2)} \\
&\leq C \sum_{|\beta|=1} \int_0^{t/2} (t-s)^{-3-\sigma} \|(-y)^\beta (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s)\|_{L^1(\mathbb{R}^2)} ds.
\end{aligned}$$

Propositions 2.9 and 3.2 lead that

$$\begin{aligned}
& \|(-y)^\beta (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)\|_{L^1(\mathbb{R}^2)} \\
&\leq \|(-y)^\beta u\|_{L^2(\mathbb{R}^2)} \|\nabla (-\Delta)^{-1} (u - MP)\|_{L^2(\mathbb{R}^2)} + M \|u - MP\|_{L^1(\mathbb{R}^2)} \|(-y)^\beta \nabla (-\Delta)^{-1} P\|_{L^\infty(\mathbb{R}^2)} \\
&\leq C s^{-\varepsilon} (1+s)^{-1+\varepsilon} \log(e+s).
\end{aligned}$$

Thus

$$\left\| \int_0^{t/2} \nabla (-\Delta)^{\sigma/2} P(t-s) * (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \right\|_{L^2(\mathbb{R}^2)} \leq C t^{-3-\sigma} (\log(e+t))^2.$$

From Lemma 2.4, we have that

$$\begin{aligned}
& \left\| \int_{t/2}^t P(t-s) * \nabla (-\Delta)^{\sigma/2} \cdot (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \right\|_{L^2(\mathbb{R}^2)} \\
&\leq C \int_{t/2}^t \{ \|\nabla (-\Delta)^{\sigma/2} u\|_{L^4(\mathbb{R}^2)} \|\nabla (-\Delta)^{-1} (u - MP)\|_{L^4(\mathbb{R}^2)} + \|u\|_{L^4(\mathbb{R}^2)} \|\nabla^2 (-\Delta)^{\frac{\sigma}{2}-1} (u - MP)\|_{L^4(\mathbb{R}^2)} \\
&\quad + \|\nabla (-\Delta)^{\sigma/2} (u - MP)\|_{L^4(\mathbb{R}^2)} \|\nabla (-\Delta)^{-1} P\|_{L^4(\mathbb{R}^2)} + \|u - MP\|_{L^4(\mathbb{R}^2)} \|\nabla^2 (-\Delta)^{\frac{\sigma}{2}-1} P\|_{L^4(\mathbb{R}^2)} \} ds.
\end{aligned}$$

Therefore, by Propositions 2.7 and 3.1 with the aid of the Sobolev inequality, and Proposition 2.10 and (1.2), we obtain that

$$\begin{aligned}
& \left\| \int_{t/2}^t P(t-s) * (-\Delta)^{\sigma/2} \nabla \cdot (u \nabla (-\Delta)^{-1} u - M^2 P \nabla (-\Delta)^{-1} P)(s) ds \right\|_{L^2(\mathbb{R}^2)} \\
&\leq C \int_{t/2}^t s^{-4-\sigma} \log(e+s) ds
\end{aligned}$$

for large t . Since

$$(-\Delta)^{\sigma/2} (J(1+t) - J(t)) = \int_0^1 \partial_t (-\Delta)^{\sigma/2} J(t+\mu) d\mu$$

and

$$\begin{aligned} \partial_t (-\Delta)^{\sigma/2} J(t) &= \nabla (-\Delta)^{\sigma/2} \cdot (P \nabla (-\Delta)^{-1} P)(t) \\ &- \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \nabla^\beta \nabla (-\Delta)^{(1+\sigma)/2} P(t-s, x-y+\lambda y) \cdot (-y)^\beta (P \nabla (-\Delta)^{-1} P)(s, y) d\lambda dy ds \\ &- \int_{t/2}^t P(t-s) * \nabla (-\Delta)^{(1+\sigma)/2} \cdot (P \nabla (-\Delta)^{-1} P)(s) ds, \end{aligned}$$

we see that

$$\|(-\Delta)^{\sigma/2} (J(1+t) - J(t))\|_{L^2(\mathbb{R}^2)} \leq C \int_0^1 (t+\mu)^{-3-\sigma} d\mu.$$

Consequently, we obtain (3.11). The Minkowski inequality and (1.2) lead (3.10) for small t . For large t , (3.10) is derived in a similar manner to above. \square

We remark that the proof for (3.10) does not require Lemma 2.4. Thus we can show (3.10) even for $p = 1$.

Proposition 3.4. *Upon the assumption of Theorem 1.4,*

$$\|\nabla (-\Delta)^{-1} (u(t) - MP(t) - m \cdot \nabla P(1+t) - M^2 J(1+t))\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-2} (1 + |\log t|)$$

for $t > 0$.

Proof. From (1.5), we see that

$$\begin{aligned} &\nabla (-\Delta)^{-1} (u(t) - MP(t) - m \cdot \nabla P(1+t) - M^2 J(1+t)) \\ &= \nabla (-\Delta)^{-1} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t)) \\ &\quad + \nabla (-\Delta)^{-1} \left(\int_0^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds - M^2 J(1+t) \right). \end{aligned}$$

We estimate the first part. Since

$$P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t) = P(t) * u_0 - MP(t) - m \cdot \nabla P(t) + m \cdot \nabla (P(t) - P(1+t)),$$

we have that

$$\begin{aligned} &\nabla (-\Delta)^{-1} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t)) \\ &= \sum_{|\alpha|=2} \int_{\mathbb{R}^2} \int_0^1 \frac{\nabla (-\Delta)^{-1} \nabla^\alpha P(t, x-y+\lambda y)}{\alpha!} (-\lambda) (-y)^\alpha u_0(y) d\lambda dy \\ &\quad + \int_0^1 \nabla (-\Delta)^{-1} (m \cdot \nabla) \partial_t P(t+\mu) d\mu \end{aligned}$$

from Taylor's theorem. Hence Lemma 2.5 yields that

$$\|\nabla (-\Delta)^{-1} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t))\|_{L^2(\mathbb{R}^2)} \leq C t^{-2}.$$

On the other hand, from

$$\begin{aligned} &P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t) \\ &= - \int_0^1 \partial_t P(t+\lambda) * u_0 d\lambda + \int_{\mathbb{R}^2} \int_0^1 \nabla P(1+t, x-y+\lambda y) \cdot (-y) u_0(y) d\lambda dy - M \int_0^1 \partial_t P(t+\lambda) d\lambda, \end{aligned}$$

this part fulfills that

$$\begin{aligned} & \left\| \nabla(-\Delta)^{-1} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t)) \right\|_{L^2(\mathbb{R}^2)} \\ & \leq C \int_0^1 (t+\lambda)^{-1} d\lambda \leq C \log(1 + \frac{1}{t}). \end{aligned}$$

Therefore we obtain that

$$\left\| \nabla(-\Delta)^{-1} (P(t) * u_0 - MP(t) - m \cdot \nabla P(1+t)) \right\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-2} (1 + |\log t|).$$

For the nonlinear term, we have that

$$\begin{aligned} & \nabla(-\Delta)^{-1} \left(\int_0^t P(t-s) * \nabla \cdot (u \nabla(-\Delta)^{-1} u)(s) ds - M^2 J(1+t) \right) \\ & = \nabla^2(-\Delta)^{-1} \int_0^{t/2} \int_{\mathbb{R}^2} (P(t-s, x-y) - P(t-s, x)) \cdot (u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P)(s, y) dy ds \\ & \quad + \nabla^2(-\Delta)^{-1} \int_{t/2}^t P(t-s) * (u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P)(s) ds \\ & \quad + M^2 \nabla(-\Delta)^{-1} (J(t) - J(1+t)). \end{aligned}$$

By Lemma 2.5 with Taylor's theorem, we obtain that

$$\begin{aligned} & \left\| \nabla(-\Delta)^{-1} \left(\nabla \cdot \int_0^t P(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds - M^2 J(1+t) \right) \right\|_{L^2(\mathbb{R}^2)} \\ & \leq C \sum_{|\beta|=1} \int_0^{t/2} (t-s)^{-2} \| (-y)^\beta (u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P)(s) \|_{L^1(\mathbb{R}^2)} ds \\ & \quad + C \int_{t/2}^t (t-s)^{-1/3} \| (u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P)(s) \|_{L^{3/2}(\mathbb{R}^2)} ds + Ct^{-2} \\ & \leq C \int_0^{t/2} (t-s)^{-2} s^{-\varepsilon} (1+s)^{-1+\varepsilon} \log(e+s) ds + C \int_{t/2}^t (t-s)^{-1/3} s^{-8/3} ds + Ct^{-2} \\ & \leq Ct^{-2} \log(e+t). \end{aligned}$$

Therefore we complete the proof. \square

Since

$$\begin{aligned} & u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P) \\ & = u \nabla(-\Delta)^{-1} (u - MP - m \cdot \nabla P - M^2 J) + M (u - MP - m \cdot \nabla P - M^2 J) \nabla(-\Delta)^{-1} P \\ & \quad + (u - MP) \nabla(-\Delta)^{-1} (m \cdot \nabla) P + M^2 (u - MP) \nabla(-\Delta)^{-1} J, \end{aligned}$$

we see (1.13) from Propositions 3.3 and 3.4.

Proof of Theorem 1.4. The decay of the first term on the right hand side of (1.5) is treated by Lemma 2.6. We divide the second term as

$$\begin{aligned}
& \int_0^t \nabla P(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds \\
&= M^2 J(t) + \int_0^t \nabla P(t-s) * \{ u \nabla(-\Delta)^{-1} u(s) - M^2 P \nabla(-\Delta)^{-1} P(s) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s) \} ds \\
&+ M \int_0^t \nabla P(t-s) * (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s) ds \\
&+ M^3 \int_0^t \nabla P(t-s) * (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s) ds.
\end{aligned}$$

Since $\int_{\mathbb{R}^2} u \nabla(-\Delta)^{-1} u dy = \int_{\mathbb{R}^2} P \nabla(-\Delta)^{-1} P dy = \int_{\mathbb{R}^2} (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) dy = \int_{\mathbb{R}^2} (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) dy = 0$, we see that

$$\begin{aligned}
& \int_0^t \nabla P(t-s) * \{ u \nabla(-\Delta)^{-1} u(s) - M^2 P \nabla(-\Delta)^{-1} P(s) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s) \} ds \\
&= \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s, y) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s, y) \} dy ds + \rho_1(t) + \rho_2(t) + \rho_3(t),
\end{aligned}$$

where

$$\begin{aligned}
\rho_1(t) &= \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t, x) (-y)^\beta \right) \\
&\quad \cdot \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s, y) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s, y) \} dy ds, \\
\rho_2(t) &= \int_{t/2}^t P(t-s) * \nabla \cdot \{ u \nabla(-\Delta)^{-1} u(s) - M^2 P \nabla(-\Delta)^{-1} P(s) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s) \} ds, \\
\rho_3(t) &= - \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_{t/2}^\infty \int_{\mathbb{R}^2} (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\
&\quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla) P + (m \cdot \nabla) P \nabla(-\Delta)^{-1} P) (1+s, y) \\
&\quad - M^3 (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P) (1+s, y) \} dy ds.
\end{aligned}$$

Moreover, from $\int_{\mathbb{R}^2} (-y)^\beta (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) dy = 0$ for $|\beta| \leq 1$, we obtain that

$$\begin{aligned} & \int_0^t \nabla P(t-s) * (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (1+s) ds \\ &= \int_0^t \nabla P(t-s) * (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (s) ds + \rho_4(t) + \rho_5(t), \end{aligned}$$

where

$$\begin{aligned} \rho_4(t) &= \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\ &\quad \cdot \left\{ (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (1+s, y) \right. \\ &\quad \left. - (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (s, y) \right\} dy ds, \\ \rho_5(t) &= \int_{t/2}^t \int_0^1 P(t-s) * \nabla \cdot \left\{ (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (1+s) \right. \\ &\quad \left. - (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P) (s) \right\} ds. \end{aligned}$$

Similarly we have that

$$\begin{aligned} & \int_0^t \nabla P(t-s) * (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s) ds \\ &= \sum_{|\beta|=1} \nabla^\beta \nabla P(t, x) \cdot \int_0^{t/2} \int_{\mathbb{R}^2} (-y)^\beta (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s, y) dy ds \\ &\quad + \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t, x) (-y)^\beta \right) \\ &\quad \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s, y) dy ds \\ &\quad + \int_{t/2}^t P(t-s) * \nabla \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s) ds \\ &= \sum_{|\beta|=1} \nabla^\beta \nabla P(t, x) \cdot \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}^2} (-y)^\beta (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1, y) dy \\ &\quad + \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x) \right) \\ &\quad \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (s, y) dy ds \\ &\quad + \int_{t/2}^t P(t-s) * \nabla \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (s) ds + \rho_6(t) + \rho_7(t), \end{aligned}$$

where

$$\begin{aligned} \rho_6(t) &= \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t, x) (-y)^\beta \right) \\ &\quad \cdot ((P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s, y) - (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (s, y)) dy ds, \\ \rho_7(t) &= \int_{t/2}^t P(t-s) \\ &\quad * \nabla \cdot ((P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (1+s) - (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P) (s)) ds. \end{aligned}$$

Now we remark that

$$\sum_{|\beta|=1} \nabla^\beta \nabla P(t, x) \cdot \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}^2} (-y)^\beta (P \nabla(-\Delta)^{-1} J + J \nabla(-\Delta)^{-1} P)(1, y) dy = K(t)$$

since $P \partial_j(-\Delta)^{-1} J + \partial_j(-\Delta)^{-1} P$ is an odd function in x_j and is an even function in another spatial variable. Consequently, we see that

$$\begin{aligned} & \int_0^t \nabla P(t-s) * (u \nabla(-\Delta)^{-1} u)(s) ds = M^2 J(t) + M^3 K(t) + J_2(t) \\ (3.12) \quad & + \sum_{|\beta|=1} \nabla^\beta \nabla P(t) \cdot \int_0^\infty \int_{\mathbb{R}^2} (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\ & - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)(1+s, y) \} dy ds \\ & + \rho_1(t) + \dots + \rho_7(t). \end{aligned}$$

We can show that

$$(3.13) \quad \|\rho_1(t)\|_{L^q(\mathbb{R}^2)} + \|\rho_2(t)\|_{L^q(\mathbb{R}^2)} + \|\rho_3(t)\|_{L^q(\mathbb{R}^2)} = o(t^{-2(1-\frac{1}{q})-2})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$ from the similar way as in the proof of Theorem 1.2. Indeed, we divide ρ_1 into $\rho_1 = \rho_{1,1} + \rho_{1,2}$, where

$$\begin{aligned} \rho_{1,1}(t) &= \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\ & \quad \cdot \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\ & \quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)(1+s, y) \} dy ds, \\ \rho_{1,2}(t) &= \sum_{|\beta|=1} \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla^\beta \nabla P(t-s, x) - \nabla^\beta \nabla P(t, x)) \\ & \quad \cdot (-y)^\beta \{ u \nabla(-\Delta)^{-1} u(s, y) - M^2 P \nabla(-\Delta)^{-1} P(s, y) \\ & \quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)(1+s, y) \} dy ds. \end{aligned}$$

We consider only $\rho_{1,1}$, and split it as

$$\begin{aligned} \rho_{1,1}(t) &= \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\ & \quad \cdot u(s) \nabla(-\Delta)^{-1} (u(s) - MP(s) - m \cdot \nabla P(1+s) - M^2 J(1+s)) dy ds \\ & + M \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\ & \quad \cdot (u(s) - MP(s) - m \cdot \nabla P(1+s) - M^2 J(1+s)) \nabla(-\Delta)^{-1} P(s) dy ds \\ & + \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\ & \quad \cdot (u(s) - MP(s)) \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J)(1+s) dy ds. \end{aligned}$$

The similar procedure as in the proof of Theorem 1.2 with the aid of Propositions 3.3 and 3.4 leads that

$$\begin{aligned}
& \left\| \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \right. \\
& \quad \cdot u(s) \nabla(-\Delta)^{-1} (u(s) - MP(s) - m \cdot \nabla P(1+s) - M^2 J(1+s)) dy ds \Big\|_{L^q(\mathbb{R}^2)} \\
& + \left\| \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \right. \\
& \quad \cdot (u(s) - MP(s) - m \cdot \nabla P(1+s) - M^2 J(1+s)) \nabla(-\Delta)^{-1} P(s) dy ds \Big\|_{L^q(\mathbb{R}^2)} = o(t^{-2(1-\frac{1}{q})-2})
\end{aligned}$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Taylor's theorem provides that

$$\begin{aligned}
& \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \\
& \quad \cdot (u(s) - MP(s)) \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) (1+s) dy ds \\
& = \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \\
& \quad \cdot \lambda (-y)^\beta (u(s) - MP(s)) \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) (1+s) d\lambda dy ds.
\end{aligned}$$

Thus, by Lemma 2.5 and Proposition 2.10, we have that

$$\begin{aligned}
& \left\| \int_0^{t/2} \int_{\mathbb{R}^2} \left(\nabla P(t-s, x-y) - \sum_{|\beta| \leq 1} \nabla^\beta \nabla P(t-s, x) (-y)^\beta \right) \right. \\
& \quad \cdot (u(s) - MP(s)) \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) (1+s) dy ds \Big\|_{L^q(\mathbb{R}^2)} \\
& \leq C \int_0^{t/2} (t-s)^{-2(1-\frac{1}{q})-3} \|u(s) - MP(s)\|_{L^1(\mathbb{R}^2)} \| |y|^2 \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) (1+s) \|_{L^\infty(\mathbb{R}^2)} ds \\
& \leq C \int_0^{t/2} (t-s)^{-2(1-\frac{1}{q})-3} (1+s)^{-1} \log(e+s) ds = o(t^{-2(1-\frac{1}{q})-2}).
\end{aligned}$$

Here we used the relation $\sup_{s>0} \| |y|^2 \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) (1+s) \|_{L^\infty(\mathbb{R}^2)} < \infty$. Indeed, since

$$\nabla(-\Delta)^{-1} J(1) = \sum_{j=1}^2 \int_0^1 \nabla \partial_j (-\Delta)^{-1} P(1-s) * (P \partial_j (-\Delta)^{-1} P)(s) ds,$$

we see from the Hörmander-Mikhlin-type estimate that

$$|\nabla(-\Delta)^{-1} J(1)| \leq C (1+|y|^2)^{-1}.$$

A coupling of this and (1.7) yields that $\sup_{s>0} \| |y|^2 \nabla(-\Delta)^{-1} J(1+s) \|_{L^\infty(\mathbb{R}^2)} < \infty$. Analogously we obtain that $\sup_{s>0} \| |y|^2 \nabla^2(-\Delta)^{-1} P(1+s) \|_{L^\infty(\mathbb{R}^2)} < \infty$. Similarly, we can treat ρ_2 , and confirm that

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^2} |y_j (u \nabla(-\Delta)^{-1} u - M^2 P \nabla(-\Delta)^{-1} P \\
& \quad - M (P \nabla(-\Delta)^{-1} (m \cdot \nabla P + M^2 J) + (m \cdot \nabla P + M^2 J) \nabla(-\Delta)^{-1} P)| dy ds < +\infty.
\end{aligned}$$

Moreover we have the estimate for ρ_3 . Taylor's theorem yields that

$$\begin{aligned} \rho_4(t) = & \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \lambda(-y)^\beta \\ & \cdot \partial_t (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P)(s+\mu, y) d\mu d\lambda dy ds \end{aligned}$$

and

$$\rho_5(t) = \int_{t/2}^t \int_0^1 P(t-s) * \nabla \cdot \partial_t (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P)(s+\mu, y) d\mu ds.$$

For $1 \leq q \leq \infty$, we see from Lemma 2.5 that

$$\begin{aligned} (3.14) \quad & \|\rho_4(t)\|_{L^p(\mathbb{R}^2)} + \|\rho_5(t)\|_{L^q(\mathbb{R}^2)} \\ & \leq C \int_0^{t/2} \int_0^1 (t-s)^{-2(1-\frac{1}{q})-3} (s+\mu)^{-1} d\mu ds + C \int_{t/2}^t \int_0^1 (s+\mu)^{-2(1-\frac{1}{q})-4} d\mu ds \\ & = o(t^{-2(1-\frac{1}{q})-2}) \end{aligned}$$

as $t \rightarrow \infty$. Analogously

$$(3.15) \quad \|\rho_6(t)\|_{L^q(\mathbb{R}^2)} + \|\rho_7(t)\|_{L^q(\mathbb{R}^2)} = o(t^{-2(1-\frac{1}{q})-2})$$

as $t \rightarrow \infty$ for $1 \leq q \leq \infty$. Applying (3.13)-(3.15) to (3.12), we complete the proof. \square

3.5. Properties of the correction terms. Before closing this paper, we confirm the basic properties of the correction terms in the theorems.

Proposition 3.5. *The function \tilde{J} in (1.9) satisfies (1.10).*

Proof. It suffices to show that the first term on \tilde{J} is well-defined. Since $\int_{\mathbb{R}^3} P\nabla(-\Delta)^{-1}P dy = 0$, we see from Taylor's theorem that

$$\begin{aligned} & \int_0^{t/2} \int_{\mathbb{R}^3} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \cdot (P\nabla(-\Delta)^{-1}P)(s, y) dy ds \\ & = \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^3} \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \lambda \cdot (-y)^\beta (P\nabla(-\Delta)^{-1}P)(s, y) d\lambda dy ds. \end{aligned}$$

Hence Lemma 2.5 leads that

$$\begin{aligned} & \left\| \int_0^{t/2} \int_{\mathbb{R}^3} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \cdot (P\nabla(-\Delta)^{-1}P)(s, y) dy ds \right\|_{L^p(\mathbb{R}^3)} \\ & \leq C \int_0^{t/2} (t-s)^{-3(1-\frac{1}{p})-3} \| |y|^2 (P\nabla(-\Delta)^{-1}P)(s, y) \|_{L^1(\mathbb{R}^3)} ds \\ & \leq C \int_0^{t/2} (t-s)^{-3(1-\frac{1}{p})-3} ds \leq C t^{-3(1-\frac{1}{p})-2} \end{aligned}$$

for $1 \leq p \leq \infty$ and $t > 0$. Thus $\tilde{J} \in C((0, \infty), L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$. We see that $\lambda^5 \tilde{J}(\lambda t, \lambda x) = \tilde{J}(t, x)$ for any $\lambda > 0$. Particularly $\tilde{J}(t, x) = t^{-5} \tilde{J}(1, t^{-1}x)$ and we obtain the second assertion. \square

Proposition 3.6. *The function J_2 defined by (1.11) satisfies (1.12).*

Proof. Since $\nabla(-\Delta)^{-1}$ is skew adjoint in $L^2(\mathbb{R}^2)$, we see that

$$\int_{\mathbb{R}^2} (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P)(s, y) dy = 0.$$

Moreover,

$$\int_{\mathbb{R}^2} y_j (P\nabla(-\Delta)^{-1}(m \cdot \nabla)P + (m \cdot \nabla)P\nabla(-\Delta)^{-1}P)(s, y) dy = 0$$

since $y_j(P\nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P)\nabla(-\Delta)^{-1}P)(s, y)$ is an odd function in y_1 or y_2 . Hence Taylor's theorem says that

$$\begin{aligned} & \int_0^{t/2} \nabla P(t-s) * (P\nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P)\nabla(-\Delta)^{-1}P)(s) ds \\ &= \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \lambda \\ & \quad \cdot (-y)^\beta (P\nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P)\nabla(-\Delta)^{-1}P)(s, y) d\lambda dy ds. \end{aligned}$$

Thus we see from Lemma 2.5 that

$$\begin{aligned} & \left\| \int_0^{t/2} \nabla P(t-s) * (P\nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P)\nabla(-\Delta)^{-1}P)(s) ds \right\|_{L^p(\mathbb{R}^2)} \\ & \leq C \int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-3} \| |y|^2 (P\nabla(-\Delta)^{-1}(m \cdot \nabla P) + (m \cdot \nabla P)\nabla(-\Delta)^{-1}P)(s) \|_{L^1(\mathbb{R}^2)} ds \\ & \leq C \int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-3} ds \leq Ct^{-2(1-\frac{1}{p})-2} \end{aligned}$$

for $1 \leq p \leq \infty$. In a similar procedure as in the proof of Proposition 3.5, we see that

$$\begin{aligned} & \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \\ & \quad \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P)(s, y) dy ds \\ &= \sum_{|\beta|=2} \int_0^{t/2} \int_{\mathbb{R}^2} \int_0^1 \frac{\nabla^\beta \nabla P(t-s, x-y+\lambda y)}{\beta!} \lambda \\ & \quad \cdot (-y)^\beta (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P)(s, y) d\lambda dy ds \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) + (y \cdot \nabla) \nabla P(t, x)) \right. \\ & \quad \cdot (P\nabla(-\Delta)^{-1}J + J\nabla(-\Delta)^{-1}P)(s, y) dy ds \left. \right\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})-2}. \end{aligned}$$

Therefore J_2 is well-defined in $C((0, \infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$. The scaling-properties of P say that $J_2(t, x) = t^{-4}J_2(1, t^{-1}x)$. Hence we get the second assertion of (1.12). \square

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